Proof:

We aim to prove that if $-x \mod 3 = 1$ and x is odd, then the expression

$$\frac{x \cdot 2^{2n-1} - 1}{3}$$

evaluates to a natural number ($\in \mathbb{N}$) for all $n \in \mathbb{N}$.

Step 1: Analyze the condition $-x \mod 3 = 1$

The condition $-x \mod 3 = 1$ implies:

$$-x \equiv 1 \pmod 3.$$

This is equivalent to:

$$x \equiv -1 \pmod 3 \quad \text{or} \quad x \equiv 2 \pmod 3.$$

Additionally, it is given that x is odd. This ensures that x is in the subset of odd integers satisfying $x \equiv 2 \pmod 3$. Thus:

$$x = 3k + 2, \quad \text{for some integer } k, \quad \text{and } x \text{ is odd.}$$

Step 2: Substitute x into the expression

Substitute $x = 3k + 2$ into the given expression:

$$\frac{x \cdot 2^{2n-1} - 1}{3}.$$

This becomes:

$$\frac{(3k+2) \cdot 2^{2n-1} - 1}{3}.$$

Simplify:

$$\frac{3k \cdot 2^{2n-1} + 2 \cdot 2^{2n-1} - 1}{3}.$$

Step 3: Break into terms divisible by 3

Split the expression as follows:

$$\frac{3k \cdot 2^{2n-1}}{3} + \frac{2 \cdot 2^{2n-1} - 1}{3}.$$

The first term, $\frac{3k \cdot 2^{2n-1}}{3}$, is clearly an integer since $3k \cdot 2^{2n-1}$ is divisible by 3.

Now consider the second term:

$$\frac{2 \cdot 2^{2n-1} - 1}{3}.$$

Step 4: Show divisibility of $2 \cdot 2^{2n-1} - 1$ by 3

Let $y = 2^{2n-1}$. Then:

$$2 \cdot 2^{2n-1} - 1 = 2y - 1.$$

Behavior of $2^{2n-1} \mod 3$: The powers of 2 modulo 3 cycle: - $2 \mod 3 = 2$, - $2^2 \mod 3 = 1$, - $2^3 \mod 3 = 2$, - $2^4 \mod 3 = 1$, and so on.

Since $2n - 1$ is odd, we have:

$$2^{2n-1} \mod 3 = 2.$$

Substitute:

$$2y - 1 \equiv 2 \cdot 2 - 1 \pmod 3 \equiv 4 - 1 \pmod 3 \equiv 3 \pmod 3.$$

Thus, $2y - 1$ is divisible by 3.

Step 5: Show Positivity

From Step 4, the numerator of the expression is $x \cdot 2^{2n-1} - 1$. Substituting $x = 3k + 2$ and $2^{2n-1} > 0$ for $n \in \mathbb{N}$, the numerator is:

$$x \cdot 2^{2n-1} - 1 > 0 \quad \text{since } x > 0 \text{ and } 2^{2n-1} > 0.$$

Thus:

$$\frac{x \cdot 2^{2n-1} - 1}{3} > 0.$$

Step 6: Conclusion

Both terms in the expression are positive integers:

$$\frac{3k \cdot 2^{2n-1}}{3} \in \mathbb{N}, \quad \frac{2 \cdot 2^{2n-1} - 1}{3} \in \mathbb{N}.$$

Therefore:

$$\frac{x \cdot 2^{2n-1} - 1}{3} \in \mathbb{N}.$$

This holds for all $n \in \mathbb{N}$, as required. Thus, the given expression is always a natural number for all $n \in \mathbb{N}$ when x is odd and satisfies $-x \mod 3 = 1$.